Fractal nearly tri-bimaximal neutrino mixing and charge-parity violation

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Developing a theory that can describe everything in the universe is of great interest, and is closely relevant to M-theory, neutrino oscillation and charge-parity (CP) violation. Although M-theory is claimed as a grand unified theory, it has not been tested by any direct experiment. Here we show that existing neutrino oscillation experimental data supports one kind of high dimensional unified theory, such as M-theory. We propose a generalization of the tri-bimaximal neutrino mixing ansatz, and we find that the latest neutrino oscillation experimental data constraints dimension in a range between 10.46 and 12.93 containing 11, which is an important prediction of M-theory. This ansatz naturally incorporates the fractal feature of the universe and leptonic CP violation into the resultant scenario of *fractal* nearly tri-bimaximal flavor mixing. We also analyze the consequences of this new ansatze on the latest experimental data of neutrino oscillations, and this theory matches the experimental data. Furthermore, an approach to construct lepton mass matrices in fractal universe under permutation symmetry is discussed. The proposed theory opens an unexpected window on the physics beyond the Standard Model.

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Introduction.– M-theory, one of the most promising theories beyond the Standard Model, is suffering from pseudoscience questions [1] because of the lack of direct experimental evidence, and causes wide discussions [2]. Recently acquired Neutrino oscillation experimental data might provide promising chances to either support or decline the M-theory. However, the relationship between neutrino oscillation and M-theory has not been fully established yet. This is because the dimensions of these two theories are not identical. Neutrino theory is a low-dimensional theory while the M-theory is 11-dimensional [3, 4]. Usually, the high-energy M-theory has to be shrunk to 4 dimensions, forming a low-energy theory to match the experimental data such as those from Large Hadron Collider (LHC). However, none of these predictions have been supported by the LHC data yet because the low-dimension M-theory has not been completely developed.

It is well known that the mixing factors of solar, atmospheric and CHOOZ neutrino oscillations read:

$$\begin{aligned} \sin^2 2\theta_{sun} &= 4\left|V_{e1}\right|^2\left|V_{e2}\right|^2 , \\ \sin^2 2\theta_{atm} &= 4\left|V_{\mu 3}\right|^2\left(1-\left|V_{\mu 3}\right|^2\right) , \\ \sin^2 2\theta_{chz} &= 4\left|V_{e3}\right|^2\left(1-\left|V_{e3}\right|^2\right) , \end{aligned} \tag{1}$$

where V is the 3×3 lepton flavor mixing matrix linking the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$. As current experimental data favor $\sin^2 2\theta_{chz} \ll \sin 2\theta_{sun} \sim \sin 2\theta_{atm} \sim o(1)$, two large flavor mixing angles can be concluded from the above equations in a specific parametrization of V: one between the 2nd and 3rd lepton families and the other between the 1st and 2nd lepton families. We pay close attention to the ansatz of "tri-bimaximal" flavor mixing pattern proposed by Harrison, Perkins and Scott [5, 6]. It predicts $\sin^2 2\theta_{sun} = 8/9$ and $\sin^2 2\theta_{atm} = 1$, consistent with the large-angle MSW [7, 8] solution to the solar neutrino problem and the atmospheric neutrino oscillation data. However, it also leads to $\sin^2 2\theta_{chz} = 0$, implying the absence of both intrinsic charge-parity (CP) violation and high-energy matter resonances in neutrino oscillations. Xing [9] discussed two possibilities to modify the tri-bimaximal neutrino mixing pattern, which can naturally incorporate CP violation into the resultant scenarios of nearly tri-bimaximal flavor mixing. Xing proposes one scenario whose predictions of $\sin^2 2\theta_{sun}$ and $\sin^2 2\theta_{atm}$ are in good agreement with the current neutrino oscillation data. However, the prediction of $\sin^2 2\theta_{chz} \approx 0.01$ is not consistent with the current data [10]: $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$.

Here, we expand the neutrino oscillation theory to 11-dimension using nonextensive statistics [11, 12]. This method has succeeded in many fields such as generalizations of relativistic and quantum equations [13], transverse momenta distributions at LHC experiments [14], dissipative optical lattices [15], plasmas [16], etc (see http://tsallis.cat.cbpf.br/biblio.htm, for a regularly updated bibliography). Nonextensive statistics is based on the fractal principle [17]. We bring it to modify the tri-bimaximal neutrino mixing pattern, which allows to incorporate CP violation and the fractal feature of the universe into the resultant scenario of fractal nearly tri-bimaximal flavor mixing. Results show that the dimension of a neutrino system using the nonextensive statistics is located between 10.46 and 12.93, which well covers the 11 predicted by the M-theory.

Constraints on dimension and mixing factors.– In order to obtain the dimension range of neutrino system, we analyze the latest neutrino oscillation experimental data with fractal nearly tri-bimaximal neutrino mixing theory

proposed in this paper (see the Supplemental Material [18]). In detail, adopting theoretical formula (see Eq. (15) in Supplemental Material [18]) $\sin^2 2\theta_{chz} = 1 - c^4$, in which $c \equiv \cos_q \theta$, combining with experimental data [10] $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, we obtain the allowed range of space-time dimension (there is an intimate relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one [19]): $10.46 \leq q \leq 12.93$. This moment, theoretical formula and experimental data have no limit on ϕ which is the source of leptonic CP violation in neutrino oscillations, so there is a set $S_{chz,q} = \{10.46 \leq q \leq 12.93, -\infty < \phi < +\infty\}$, which can be expressed in Fig. 1 with the red strip area.

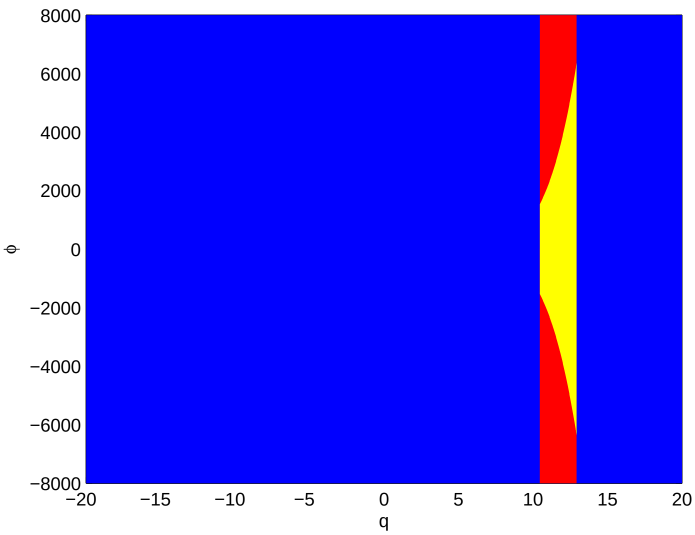

FIG. 1. The space of dimension q and phase ϕ. The figure expresses three sets and relationship among them: $S_{chz,q} = \{10.46 \leq q \leq 12.93, -\infty < \phi < +\infty\}, S_{atm,q} = \{-\infty < q < +\infty, -\infty < \phi < +\infty\} \supset S_{chz,q}, S_{sun,q}(\phi) \subset S_{chz,q}$, where, the range of q in set $S_{chz,q}$ is obtained based on experimental data $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, and ϕ is not limited now, so it can take any real number. $S_{atm,q}$ is decided by experimental data $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy, and after checked there is $S_{atm,q} \supset S_{chz,q}$. In fact, q in $S_{atm,q}$ can take all real number due to the fact that the experimental upper and lower limits of $\sin^2(2\theta_{23})$ are automatically satisfied. For upper limit, $\sin^2 2\theta_{atm} = 1 - s^4 \leq 1, \forall q \in \mathbb{R}$, and for lower limit one has $\sin^2 2\theta_{atm} \geq 0.99997675$, which satisfies the lower limit. At this moment ϕ is also not limited, so it can take any real number too On the basis of meet the above conditions $S_{sun,q}(\phi)$ is decided by experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$, So $S_{sun,q}(\phi) \subset (S_{chz,q} \cap S_{atm,q})$. Therefore, the value of q in $S_{sun,q}(\phi)$ is $\{10.46 \leq q \leq 12.93\}$, and the value of ϕ is decided by experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$, in fact only by $\sin^2 2\theta_{sun} \leq 0.867$. $S_{sun,q}(\phi)$ is expressed by the yellow area in the figure.

For sake of seeing the limit of theoretical formula (see Eq. (15) in Supplemental Material [18]) $\sin^2 2\theta_{atm} = 1 - s^4$, in which $s \equiv \sin_q \theta$, and experimental data [10] $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy on the range of q and ϕ, we do the corresponding calculation and find that q can take any real number, namely, $-\infty < q < +\infty$, and this moment, theoretical formula and experimental data also have no limit on ϕ. So, there is a set $S_{atm,q} = \{-\infty < q < +\infty, -\infty < \phi < +\infty\}$ which can be expressed in Fig. 1 with the blue strip area. There is relationship $S_{atm,q} \supset S_{chz,q}$, seeing Fig. 1. Take the intersection of these two sets we conclude that the range of space-time dimension that our theory combined with the latest neutrino oscillation experimental data allowed is between 10.46 and 12.93 containing 11, which is an important prediction of M-theory. We can also see that the allowed range of space-time dimension will be further restricted with the improvement of the experimental accuracy. The neutrino oscillation experimental data becomes the first evidence of M-theory, which will effectively eliminate the people's question to M-theory [1].

With the purpose of obtaining the range of ϕ, we adopt theoretical formula (see Eq. (15) in Supplemental Material [18]) $\sin^2 2\theta_{sun} = \frac{8}{9}\left(1 - \frac{3}{4}s^2 - sc\cos_q \phi + \frac{3}{2}s^3 c\cos_q \phi - 2s^2c^2\cos_q^2 \phi\right)$ to analyze the experimental data [10] $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$. We using numerical calculation find that the range of ϕ is depending on parameter q. The top and bottom limit of ϕ under the typical q values are in Table 1. The ϕ set under the q that allowed by all theoretical formula and experimental data, $S_{sun,q}(\phi)$, can be expresses with the yellow area in Fig. 1, and there is relationship $S_{sun,q}(\phi) \subset S_{chz,q}$.

q	$\phi_{\min}$	$\phi_{\max}$	$J_{\min}$	$J_{\max}$
1.00	0.485366	1.27256	0.0054	0.0110
10.46	-1537.79	1537.79	-0.0012	0.0012
11.00	-2162.81	2162.81	-0.0011	0.0011
12.93	-6372.47	6372.47	-0.0009	0.0009

TABLE I. The range of ϕ and strength of CP or T violation. $q = 1$ is the ideal one-dimensional case; $q = 10.46$ and 12.93 are dimension lower and upper limits allowed by existing neutrino oscillation experimental data, respectively; $q = 11$ is the prediction of M-theory. After the dimension increased, the range of phase spanned from $0.485366 \leq \phi_{q=1} \leq 1.27256$ to $-2162.81 \leq \phi_{q=11} \leq 2162.81$, increasing 3 orders; The predicted strength of CP violation is $-0.0011 \leq J_{q=11} \leq 0.0011$, which can be determined by the T- or CP-violating asymmetry in a long-baseline neutrino oscillation experiment.

In conclusion, the set of q and ϕ allowed by theoretical formula and experimental data is the intersection of sets $S_{chz,q}$, $S_{atm,q}$ and $S_{sun,q}(\phi)$, namely, $S_{sun,q}(\phi)$, i.e. the yellow area in Fig. 1.

Change on mixing factors.– Next, we investigate the change of range of ϕ after the dimension increased with the theoretical formula (see Eq.

([15] in Supplemental Material [18]) $\sin^2 2\theta_{sun} = \frac{8}{9}\left(1 - \frac{3}{4}s^2 - sc\cos_q\phi + \frac{3}{2}s^3c\cos_q\phi - 2s^2c^2\cos_q^2\phi\right)$ under the cases of $q = 1$ and $10.46 \leq q \leq 12.93$, respectively. From Table 1 and Fig. 2 we find that when $q = 1$, $0.485366 \leq \phi_{q=1} \leq 1.27256$ the order of magnitude is 1; but when $10.46 \leq q \leq 12.93$, the range of ϕ increases with the increase of q namely, from $-1537.79 \leq \phi_{q=10.46} \leq 1537.79$ to $-6372.47 \leq \phi_{q=12.93} \leq 6372.47$ with the order of magnitude 10^3. So, the order of magnitude of ϕ range increases 3 order after the dimension increased, which eliminates the question of small range of ϕ values. Specifically, when $q = 11$, $-2162.81 \leq \phi_{q=11} \leq 2162.81$.

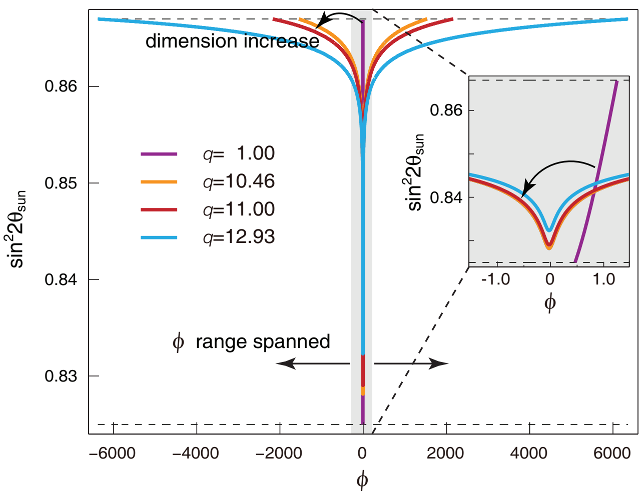

FIG. 2. The mixing factors $\sin^2 2\theta_{sun}$ against parameter ϕ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns and the top and bottom limits of experimental data. The two horizontal dotted lines are the top and bottom limits of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$. Dark red line is the limit case of $q \to 1$, and that time the experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $0.485366 \leq \phi_{q=1} \leq 1.27256$. Because the experimental data $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$ limits $10.46 \leq q \leq 12.93$, the line of $q = 1$ is not true. The orange solid line is the experimental lower limit case ($q = 10.46$), and that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-1537.79 \leq \phi_{q=10.46} \leq 1537.79$. The blue solid line is the experimental upper limit case ($q = 12.93$), and that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-6372.47 \leq \phi_{q=12.93} \leq 6372.47$. The red solid line is the M-theory predicted case ($q = 11$), that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-2162.81 \leq \phi_{q=11} \leq 2162.81$. To facilitate observing details, subgraph is the full figure's part of $-1.5 \leq \phi \leq 1.5$. The figure shows the order of magnitude of ϕ range increases 3 orders after the dimension increased.

Prediction on CP violation.– To examine the theory proposed in this paper, we give a prediction of the strength of CP or T violation in neutrino oscillations. No matter whether neutrinos are Dirac or Majorana particles, the strength of CP or T violation in neutrino oscillations is measured by a universal parameter J which is defined as [20]: $Im\left(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}\right) = J\sum_{\gamma,k}\left(\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}\right)$, in which the Greek subscripts run over (e, μ, τ), and the Latin subscripts run over $(1, 2, 3)$. Considering the lepton mixing scenario proposed above, one has (see Eq. (18) in Supplemental Material [18]): $J = \frac{1}{6}sc\sin_q\phi\left(c^2 + s^2\rho_q^2(\phi)\right)$. The prediction of the strength of CP or T violation in neutrino oscillations under typical q values are in Table 1, and especially, when $q = 11$, $-0.0011 \leq \mathrm{J}_{q=11} \leq 0.0011$. Fig. 3 expresses the prediction intuitively. The predicted strength of CP violation can be determined by the T-violating asymmetry between $\nu_\mu \to \nu_e$ and $\nu_e \to \nu_\mu$ transitions or by the CP-violating asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions in a long-baseline neutrino oscillation experiment, when the terrestrial matter effects are under control or insignificant.

Further discussions and remarks.– Our findings reveal a strong association between neutrino oscillation and M-theory at the point of 11 dimensions of space-time. This would mean that the neutrino oscillation experiment is the initial robust evidence of M-theory, broking the spell that the M-theory has no experimental evidence, eliminating pseudoscience questions [1], and opening an unexpected window on the physics beyond the Standard Model. However, we should realize that in spite of the M-theory have part truth, but not completely developed yet, and there may be other way. Fractal theory and practice [21] have illuminated that the world is of fractal. The definition of fractal dimension is more universal than the one of Euclidean dimension. Euclid dimension is just a special case of fractal dimension, and there is intimate relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one [19]. As the $q \to 1$ limit case of the fractal nearly tri-bimaximal neutrino mixing pattern under discussion, the nearly tri-bimaximal neutrino mixing pattern, as Xing [9] expected, serves as the leading-order approximation of a more complicated flavor mixing matrix (see Eq. (12) in Supplemental Material [18]), though its prediction on $\sin^2 2\theta_{chz}$ is not consistent with the experimental data. Although existing neutrino oscillation experiment data limits the range of space-time dimension between 10.46 and 12.93 (see Fig. 1), the range of space-time dimension will be narrowed down with the increasing of experimental accuracy, and we expect an exclusion of 12 dimension. In addition, we find the order of magnitude of ϕ range increases 3 orders after the dimension increased (see Fig. 2). Moreover, this theory yields a prediction (see Fig. 3) which can be determined by the T-violating asymmetry between $\nu_\mu \to \nu_e$ and $\nu_e \to \nu_\mu$ transitions or by the CP-violating asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions in a long-baseline neutrino oscillation experiment, when the terrestrial matter effects are under control or insignificant. Note that our scenario predicts that

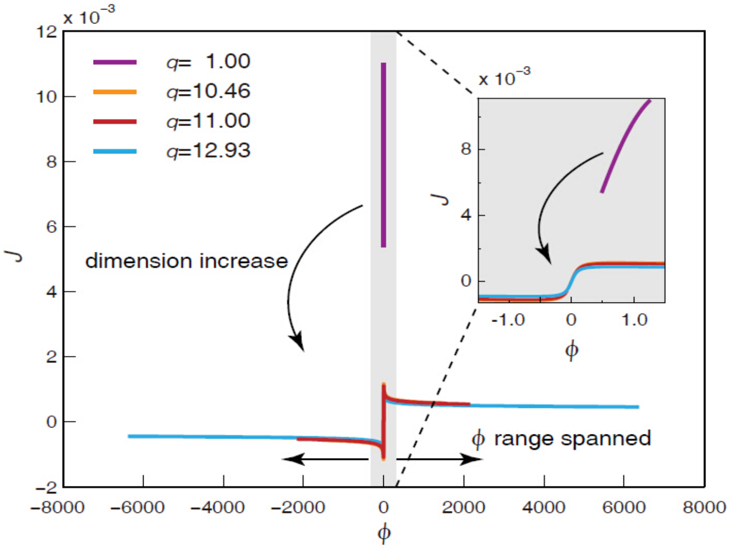

FIG. 3. The strength of CP or T violation J against parameter ϕ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns. Dark red line is the limit case of $q \to 1$, and that time the experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $0.485366 \le \phi_{q=1} \le 1.27256$. Based on the figure as well as the numerical calculations, one obtains $0.0054 \le J_{q=1} \le 0.0110$. When $q \to 1$, in theory, $\sin^2 2\theta_{chz} = 1 - \cos^4\theta = 0.0096$, which is not consistent with experimental data $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, so the line of $q = 1$ is not true. The orange solid line is the experimental lower limit case ($q = 10.46$), and that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-1537.79 \le \phi_{q=10.46} \le 1537.79$. Based on the figure as well as the numerical calculations, one obtains $-0.0012 \le J_{q=10.46} \le 0.0012$. The blue solid line is the experimental upper limit case ($q = 12.93$), and that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-6372.47 \le \phi_{q=12.93} \le 6372.47$. Based on the figure as well as the numerical calculations, one obtains $-0.0009 \le J_{q=12.93} \le 0.0009$. The red solid line is the M-theory predicted case ($q = 11$), that time the upper limit of experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ limits $-2162.81 \le \phi_{q=11} \le 2162.81$. Based on the figure as well as the numerical calculations, one obtains prediction $-0.0011 \le J_{q=11} \le 0.0011$. To facilitate observing details, subgraph is the full figure's part of $-1.5 \le \phi \le 1.5$. The figure shows the predicted strength of CP violation is $-0.0011 \le J_{q=11} \le 0.0011$, which can be determined by the T- or CP-violating asymmetry in a long-baseline neutrino oscillation experiment.

$-0.0011 \le J_{q=11} \le 0.0011$, and when $\phi = 0$, $J_{q=11} = 0$, namely, there is no CP violation. Therefore, our theory can be applied whether CP is violated or not.

Finally, let us remark that the fractal nearly tri-bimaximal mixing pattern and its possible extensions require some peculiar flavor symmetries to be imposed on the charged lepton and neutrino mass matrices. It is likely that the fractal nearly tri-bimaximal neutrino mixing pattern under discussion serves as the more complicated flavor mixing matrix that scientists are looking for [9], and one of the nearly tri-bimaximal neutrino mixing patterns is its leading-order approximation. We expect that more delicate neutrino oscillation experiments in the near future will be able to verify the fractal nearly tri-bimaximal mixing pattern, from which one may get some insight into the underlying flavor symmetry and its breaking mechanism responsible for the origin of both lepton masses and leptonic CP violation.

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Supplemental Material: Fractal nearly tri-bimaximal neutrino mixing and charge-parity violation

Fractal nearly tri-bimaximal neutrino mixing. In the picture of neutrino as Majorana particle, the light (left-handed) neutrino mass matrix M_ν must be symmetric and can be diagonalized by a single unitary transformation: $U_\nu^\dagger M_\nu U_\nu^* = Diag\{m_1, m_2, m_3\}$. In general, the charged lepton mass matrix M_l is non-Hermitian; hence, the diagonalization of M_l needs a special bi-unitary transformation: $U_l^\dagger M_l \tilde{U}_l = Diag\{m_e, m_\mu, m_\tau\}$. The lepton flavor mixing matrix V, defined to link the neutrino mass eigenstates (v_1, ν_2, ν_3) to the neutrino flavor eigenstates (v_e, ν_μ, ν_τ), measures the mismatch between the diagonalization of M_l and that of M_ν: $V = U_l^\dagger U_\nu$. It is worth noting that (m_1, m_2, m_3) and (m_e, m_μ, m_τ) are physical (real and positive) masses of light neutrinos and charged leptons, respectively.

In the flavor basis where M_l is diagonal (i.e., $U_l = \mathbf{1}$ being a unity matrix), the flavor mixing matrix is simplified to $V = U_\nu$. The tri-bimaximal neutrino mixing pattern $U_v = V_0$ can then be constructed from the product of two modified Euler rotation matrices:

$$\begin{aligned} R_{12}(\theta_x) &= \begin{pmatrix} c_x & s_x & 0 \\ -s_x & c_x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_{23}(\theta_y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_y & s_y \\ 0 & -s_y & c_y \end{pmatrix}, \end{aligned} \tag{2}$$

where $s_x \equiv \sin_q \theta_x, c_y \equiv \cos_q \theta_y$, and so on. Functions $\sin_q u$ and $\cos_q u$ can be defined with $\exp_q(u)$ which is the one-dimensional q-exponential function that naturally emerges in nonextensive statistics [11] spawned by fractal thought [17]. For a pure imaginary iu, one defines $\exp_q(iu)$ as the principal value of

$$\begin{aligned} &\exp_q(iu) = [1 + (1-q)\, iu]^{1/(1-q)}, \\ &\exp_1(iu) \equiv \exp(iu). \end{aligned} \tag{3}$$

The above function satisfies [12]:

$$\exp_q(\pm iu) = \cos_q(u) \pm i \sin_q(u), \tag{4}$$

$$\cos_q(u) = \rho_q(u) \cos\left\{\frac{1}{q-1} \arctan[(q-1)\, u]\right\}, \tag{5}$$

$$\sin_q(u) = \rho_q(u) \sin\left\{\frac{1}{q-1} \arctan[(q-1)\, u]\right\}, \tag{6}$$

$$\rho_q(u) = \left[1 + (1-q)^2 u^2\right]^{1/[2(1-q)]}, \tag{7}$$

$$\exp_q(iu) \exp_q(-iu) = \cos_q^2(u) + \sin_q^2(u) = \rho_q^2(u). \tag{8}$$

Note that $\exp_q[i(u_1+u_2)] \neq \exp_q(iu_1)\exp_q(iu_2)$ for $q \neq 1$[11]. Then we obtain:

$$\begin{aligned} V_0 &= R_{23}(\theta_y) \otimes R_{12}(\theta_x) \\ &= \begin{pmatrix} c_x & s_x & 0 \\ -s_x c_y & c_x c_y & s_y \\ s_x s_y & -s_y c_x & c_y \end{pmatrix}. \end{aligned} \tag{9}$$

The vanishing of the (1, 3) element in V_0 assures an exact decoupling between solar $(\nu_e \to v_\mu)$ and atmospheric $(\nu_\mu \to v_\tau)$ neutrino oscillations. The general form of the corresponding neutrino mass matrix M_ν is

$$\begin{aligned} M_\nu &= V_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V_0^T \\ &= \begin{pmatrix} c_x^2 m_1 + s_x^2 m_2 & -c_x c_y s_x (m_1 - m_2) & c_x s_x s_y (m_1 - m_2) \\ -c_x c_y s_x (m_1 - m_2) & c_y^2 s_x^2 m_1 + c_x^2 c_y^2 m_2 + s_y^2 m_3 & -c_y s_y \left(s_x^2 m_1 + c_x^2 m_2 - m_3\right) \\ c_x s_x s_y (m_1 - m_2) & -c_y s_y \left(s_x^2 m_1 + c_x^2 m_2 - m_3\right) & s_x^2 s_y^2 m_1 + c_x^2 s_y^2 m_2 + c_y^2 m_3 \end{pmatrix}. \end{aligned} \tag{10}$$

Taking $q = 1$, $\theta_x = \arctan\left(1/\sqrt{2}\right) \approx 35.3^o$ and $\theta_y = 45^o$, the results in usual space-time are reproduced, and M_ν might have a meaningful interpretation in an underlying theory of neutrino masses with specific flavor symmetries [9].

The tri-bimaximal neutrino mixing pattern will be modified, if U_l deviates somehow from the unity matrix. This can certainly happen, provided that the charged lepton mass matrix M_l is not diagonal in the flavor basis. As $U_\nu = V_0$ describes a product of two special Euler rotations in the real (2, 3) and (1, 2) planes, the simplest form of U_l that allows $V = U_l^\dagger U_v$ to cover the whole 3×3 space should be $U_l = R_{12}(\theta_x, q=1)$ or $U_l = R_{31}(\theta_z, q=1)$ (see Ref. [24, 25] for a detailed discussion). When $U_l = R_{31}(\theta_z, q=1)$ is adopted, the calculated result [9] $0.873 \leq sin^2 2\theta_{sun}^{(z)} \leq 0.903$ is not well consistent with the

experimental data [10] $\sin^2 2\theta_{12} = 0.846 \pm 0.021$. Therefore we focus on the calculation of case $U_l = R_{12}(\theta_x)$. For convenience, θ_x will be replaced by θ in the following context.

To make CP violation and the fractal feature of the universe be naturally incorporated into V, we adopt the following complex rotation matrices:

$$R_{12}(\theta,\phi) = \begin{pmatrix} c & s e_q^{i\phi} & 0 \\ -s e_q^{-i\phi} & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where $c \equiv \cos_q \theta$, $s \equiv \sin_q \theta$, and $e_q^{i\phi} = \exp_q(i\phi)$. In this case, we obtain the lepton flavor mixing of the following pattern:

$$\begin{aligned} V &= R_{12}^{\dagger}(\theta,\phi) \otimes V_0 \\ &= \begin{pmatrix} \frac{1}{\sqrt{6}}\left(2c + s e_q^{i\phi}\right) & \frac{1}{\sqrt{3}}\left(c - s e_q^{i\phi}\right) & -\frac{1}{\sqrt{2}} s e_q^{i\phi} \\ -\frac{1}{\sqrt{6}}\left(c - 2 s e_q^{-i\phi}\right) & \frac{1}{\sqrt{3}}\left(c + s e_q^{-i\phi}\right) & \frac{1}{\sqrt{2}} c \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned} \tag{12}$$

V represents a fractal nearly tri-bimaximal flavor mixing scenario, if the rotation angle θ is small. The parameter ϕ in V are the source of leptonic CP violation in neutrino oscillations.

Constraints on dimension, mixing factors and CP violation. The mixing angle θ is expected to be a simple function of the ratios of charged lepton masses due to the fact that it arises from the diagonalization of M_l. Then the strong mass hierarchy of the charged leptons naturally ensures the smallness of θ as we will see later.

Indeed, a proper texture of M_l which may lead to the flavor mixing pattern V is

$$M_l = \begin{pmatrix} 0 & C_l & 0 \\ C_l^* & B_l & 0 \\ 0 & 0 & A_l \end{pmatrix}, \tag{13}$$

where $A_l = m_\tau$, $B_l = m_\mu - m_e$, and $C_l = \sqrt{m_e m_\mu} e_q^{i\phi}$. Then the mixing angle θ in V reads

$$\tan_q(\theta) = \frac{\sin_q \theta}{\cos_q \theta} = \sqrt{\frac{m_e}{m_\mu}}. \tag{14}$$

It is easy to prove that when $q \to 1$, the results in usual space-time are recovered, namely [9], $C_l = \sqrt{m_e m_\mu} e^{i\phi}$, $\tan(2\theta) = 2\frac{\sqrt{m_e m_\mu}}{m_\mu - m_e}$. Given the hierarchy of three charged lepton masses (i.e., $m_e \ll m_\mu \ll m_\tau$) and $q \sim o(1)$, we have $\tan_q \theta \approx \tan\theta \approx \sin\theta \approx \sqrt{m_e/m_\mu}$ to a good degree of accuracy. Numerically, we find $\theta \approx 3.978^o$ with the inputs $m_e = 0.511$MeV and $m_\mu = 105.658$MeV [10].

In the next step we calculate the mixing factors of solar, atmospheric and reactor neutrino oscillations. According to this theory, one obtains

$$\begin{aligned} \sin^2 2\theta_{sun} &= \tfrac{8}{9}\left(1 - \tfrac{3}{4}s^2 - sc\cos_q\phi + \tfrac{3}{2}s^3 c\cos_q\phi - 2s^2c^2\cos_q^2\phi\right), \\ \sin^2 2\theta_{atm} &= 1 - s^4, \\ \sin^2 2\theta_{chz} &= 1 - c^4. \end{aligned} \tag{15}$$

Note when $q \to 1$, the results in usual space-time are recovered [9]:

$$\begin{aligned} \sin^2 2\theta_{sun} &= \tfrac{8}{9}\left(1 - \tfrac{3}{4}\sin^2\theta - \sin\theta\cos\theta\cos\phi + \tfrac{3}{2}\sin^3\theta\cos\theta\cos\phi - 2\sin^2\theta\cos^2\theta\cos^2\phi\right), \\ \sin^2 2\theta_{atm} &= 1 - \sin^4\theta, \\ \sin^2 2\theta_{chz} &= 1 - \cos^4\theta. \end{aligned} \tag{16}$$

In this scenario, adopting experimental data [10] $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, one obtains $10.46 \le q \le 12.93$; thus there is $0.999987 \le \sin^2 2\theta_{atm} \le 0.99999$, which is highly consistent with the experimental data [10]: $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy; in addition, to make $\sin^2 2\theta_{sun} \le 0.867$ to accord with the experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$, one needs only $-1537.79 \le \phi_{q=10.46} \le 1537.79$ or $-6372.47 \le \phi_{q=12.93} \le 6372.47$, which are much better than the usual space-time case $(0.485366 \le \phi_{q=1} \le 1.27256)$.

Additionally, given that q is close to 11 and the intimate relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one [19], we assume $q = 11$, then this scenario gives the predicted values of $\sin^2 2\theta_{chz} = 0.082456$ and $\sin^2 2\theta_{atm} = 0.999987$ which amazingly fit in with the current data [10] $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$ and $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy ($\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy), respectively; the range of parameter $-2162.81 \le \phi_{q=11} \le 2162.81$ limited by current data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ is also much better than that in usual space-time $(0.485366 \le \phi_{q=1} \le 1.27256)$. According to the calculations above, we come to the following conclusions: i) the universe is fractal with high dimension; ii) some high dimensional space-time theo-

ries, such as M-theory, can be in line with expectations. A numerical illustration of $\sin^2 2\theta_{sun}$ as the function of q and ϕ is shown in Fig. 2, where the two horizontal lines are the top and bottom limits of experimental data. As can be seen from the figure, in $\phi = 0$ case, $\sin^2 2\theta_{sun}$ very sensitively dependent on ϕ.

The strength of CP or T violation in neutrino oscillations, no matter whether neutrinos are Dirac or Majorana particles, is measured by a universal parameter J which is defined as [20]:

$$Im\left(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}\right) = J\sum_{\gamma,k}\left(\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}\right), \tag{17}$$

in which the Greek subscripts run over (e, μ, τ), and the Latin subscripts run over $(1, 2, 3)$. Considering the lepton mixing scenario proposed above, one has

$$J = \frac{1}{6}sc\sin_q\phi\left(c^2 + s^2\rho_q^2(\phi)\right). \tag{18}$$

Obviously, when $q \to 1$, the result in usual space-time is recovered [9]:

$$J = \frac{1}{6}sc\sin\phi. \tag{19}$$

Based on Figs. 2 and 3 as well as the numerical calculations, one obtains the table I. The strength of CP or T violation J in fractal nearly tri-bimaximal neutrino mixing patterns is predicted as: $-0.0011 \le J_{q=11} \le 0.0011$. The experimental data of strength of CP or T violation may limit the range of parameter ϕ, but unfortunately at present, there is no experimental information on the Dirac and Majorana CP violation phases in the neutrino mixing matrix is available [10]. The former could be determined by the T-violating asymmetry between $\nu_\mu \to \nu_e$ and $\nu_e \to \nu_\mu$ transitions or by the CP-violating asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions in a long-baseline neutrino oscillation experiment, when the terrestrial matter effects are under control or insignificant.